Self inductance of a wire loop as a curve integral

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Abstract

It is shown that the self inductance of a wire loop may be written as a curve integral akin to the Neumann formula for the mutual inductance of two wire loops. The only difference is that contributions where the two integration variables get too close to each other must be excluded from the curve integral and evaluated in detail. The contributions of these excluded segments depend on the distribution of the current in the cross section of the wire. They add to a simple constant proportional to the wire length. The error of the new expression is of first order in the wire radius if there are sharp corners and of second order in the wire radius for smooth wire loops.

1 Introduction

Electrical inductance plays a crucial role in power plants, transformers and electronic devices. The coefficients of self and mutual inductance required to quantitatively describe inductance belong to the field of magnetostatics. Calculating inductance coefficients with analytic techniques is, however, impossible except in simple cases. The mathematical reason for the difficulty is that the Laplace equation allows analytic solutions only for some symmetric constellations. There thus are only a few closed-form expressions for these coefficients. In practice one often is forced to use approximations, finite element methods or other numerical techniques. The situation is somewhat less gloomy in systems where the current flows in thin wires. This situation is analogous to an electrostatic system of point charges, where electric field and electrostatic energy directly follow from the given charge distribution, while in a generic system charge and current distributions also are unknown at the outset.

The purpose of this article is to derive a new expression for the self inductance of a wire loop, giving self inductance as a curve integral similar to the Neumann formula for mutual inductance. The starting point is the expression

$$W = \frac{\mu_0}{8\pi} \int \frac{\mathbf{j}\left(\mathbf{x}\right) \cdot \mathbf{j}\left(\mathbf{x}'\right)}{\left|\mathbf{x} - \mathbf{x}'\right|} d^3x d^3x' \tag{1}$$

for the magnetic field energy of a system with current density $\mathbf{j}\left(\mathbf{x}\right)$, where μ_0 is the magnetic constant.[1] This expression essentially was already given by Neumann in 1845.[2]

It resembles the expression for gravitational or electrostatic potential energy, the only new ingredient is the scalar product between the current elements.

For a current density $\mathbf{j}(\mathbf{x}) = \sum I_m \mathbf{j}_m(\mathbf{x})$ corresponding to N separate current loops with currents I_m and *normalized* current densities $\mathbf{j}_m$ it follows

$$W = \frac{\mu_0}{8\pi} \sum_{m,n=1}^{N} I_m I_n \int \frac{\mathbf{j}_m(\mathbf{x}) \cdot \mathbf{j}_n(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|} d^3x d^3x' \stackrel{!}{=} \frac{1}{2} \sum_{m,n=1}^{N} L_{m,n} I_m I_n. \tag{2}$$

If the currents flow in thin wires, then the integrals become curve integrals, and one immediately reads off the Neumann expression for mutual inductance of two (filamentary) current loops[2]

$$L_{1,2} = \frac{\mu_0}{4\pi} \oint \frac{d\mathbf{x}_1 \cdot d\mathbf{x}_2}{|\mathbf{x}_1 - \mathbf{x}_2|}. \tag{3}$$

It is plausible that there exists a similar expression for the self inductance of a wire loop, but we were not able to find any hint in the literature. Formally one might read off from equation (2) an expression similar to equation (3), where the two closed curves coincide. But this cannot be correct, because $|\mathbf{x}-\mathbf{x}'|$ now vanishes and the integral isn't defined. Instead we will proof

$$L = \frac{\mu_0}{4\pi} \left(\oint \frac{d\mathbf{x} \cdot d\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|} \right)_{|s-s'|>a/2} + \frac{\mu_0}{2\pi} lY + ... \tag{4}$$

where a denotes the wire radius and l the length of the wire. The variable s measures the length along the wire axis. The constant Y depends on the distribution of the current in the cross section of the wire: $Y = 0$ if the current flows in the wire surface, $Y = 1/4$ when the current is homogeneous across the wire. The ellipses represents terms like $O(\mu_0 a)$ and $O(\mu_0 a^2/l)$, which are negligible for $l \gg a$.

In hindsight it is completely natural to use a cutoff of order a in the curve integral. In fact, the exact value of this cutoff is arbitrary, because the contribution proportional to lY also depends on this cutoff. The simplest way to determine Y would be to compare the expression with the self inductance of a long rectangle.

2 Simple derivation

Consider equation (2) with $N = 1$ for a thin wire with circular cross section, radius a and length l. Let s denote the length along the axis of the wire. The planes perpendicular to the wire axis then define a projection from the bulk of the wire onto the axis, $\mathbf{x} \to s(\mathbf{x})$ (we won't consider here cases where the centre of wire curvature is within the wire, and this projection isn't unique any more). Then, selecting a length scale b satisfying $a \ll b \ll l$ allows to write $L = (\mu_0/4\pi)\left(\overline{L} + \widehat{L}\right)$ with

$$\begin{aligned} \overline{L} &= \left(\int \frac{\mathbf{j}(\mathbf{x})\, \mathbf{j}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|} d^3x d^3x' \right)_{|s-s'|>b}, \\ \widehat{L} &= \left(\int \frac{\mathbf{j}(\mathbf{x})\, \mathbf{j}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|} d^3x d^3x' \right)_{|s-s'|<b}. \end{aligned} \tag{5}$$

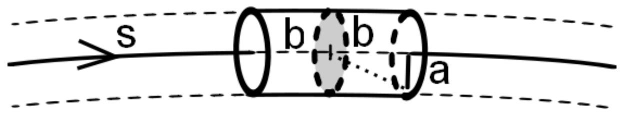

Figure 1: A section of a wire with radius a, with a segment of length $2b$, and a plane perpendicular to the wire axis at the center of the segment.

The second part contains contributions from all point pairs $\{\mathbf{x}, \mathbf{x}'\}$ with a distance along the axis smaller than b, the first the complement of this set (s is a cyclic quantity). For given $\mathbf{x}$ the planes at $s(\mathbf{x}) \pm b$ delimitate the points $\mathbf{x}'$ contributing to the first or second integral, see figure (1). $\overline{L}$ now approximately becomes a curve integral and $\widehat{L}$ essentially consists of cylinders of length $2b$.

The strategy then is to replace $\overline{L}$ with the curve integral and to explicitly evaluate the contribution of the cylinders in $\widehat{L}$. The cylinders are long in comparison to the radius because of $a \ll b$ and straight (at least most of them) because of $b \ll l$. Actually the only requirement for the lengths is $a \ll l$, the length $b = \sqrt{al}$ then satisfies $a \ll b \ll l$. The approximation thus is exact in the limit $a \ll l$ except in special cases. Inserting $\widehat{L}_0$ for a straight segment from equation (A.4) in the appendix thus leads to

$$L = \frac{\mu_0}{4\pi} \left(\oint \frac{d\mathbf{x} \cdot d\mathbf{x}'}{|\mathbf{x} - \mathbf{x}'|} \right)_{|s-s'|>b} + \frac{\mu_0 l}{2\pi} \left(\ln\left(\frac{2b}{a}\right) + Y \right) + ... \tag{6}$$

This expression cannot depend on the (more or less) arbitrary length scale b. The curve integral thus is $\overline{L}(b) = const - \frac{\mu_0 l}{2\pi} \ln(2b/b_0)$. But b is the only "short" length scale in the curve integral, and $\overline{L}(b)$ thus also is valid for $b = a/2$. The expression (6) therefore doesn't change if one formally sets $b = a/2$. Equation (6) now agrees with equation (4), but some questions remain.

First of all, how accurate is formula (4)? The curve integral is a purely geometric quantity with dimension "length" and order of magnitude l. Plausible expressions for the order of magnitude of the relative error are a/l, $(a/l)^2$ and $(a/R)^2$, with R a typical curvature radius of the wire loop. Errors of this order normally are negligible (and also occur in the Neumann formula for mutual inductance). But the derivation of formula (4) is not as straightforward, and so what are the actual limits or exceptions?

3 Examples and comparison with exact self inductance

To get an impression of the accuracy we have compared self-inductances calculated with the curve integral with the result of a numeric evaluation of the nominally 6-dimensional integral in equation (2). This integral becomes 4-dimensional if the currents flow in the wire surface (skin effect, $Y = 0$), and the results below correspond to this skin effect case. The order of magnitudes of the error terms identified here are corroborated below in a more detailed derivation of formula (4).

Straight segment

The first example is a straight segment with length c and complete skin effect. This of course isn't a closed circuit, but it might be an edge of a rectangle. The orthogonal edges of the rectangle don't interact with the segment because of the scalar product $\mathbf{j} \cdot \mathbf{j}'$. What is missing for a rectangle are the interaction terms with the opposite edges (and the small contributions from the corners). In this case the volume integral (2) may even be evaluated analytically with the result

$$L = \frac{\mu_0}{4\pi} \left\{ 2c \left[\ln\left(\frac{2c}{a}\right) - 1 \right] + 8a/\pi - a^2/c + ... \right\},$$

while the curve integral (4) leads to

$$L_c(c) = \frac{\mu_0}{4\pi} \left\{ 2c \left[\ln\left(\frac{2c}{a}\right) - 1 \right] + a \right\}. \tag{7}$$

The difference is of order $O(\mu_0 a)$, much smaller than $\mu_0 c$ for $c \gg a$.

Circular loop

The next example is a ring with radius R. The curve integral (4) gives

$$L_c = \mu_0 R \left(\ln(8R/a) - 2 + Y\right) + \mu_0 O\left(a^2/R\right).$$

This expression also may be found in the literature, derived with the help of elliptic functions and some approximations in a much more complicated way. The table displays the ratio of the exact inductance and L_c for some values of R/a,

R/a	1	2	3	5	10	...
L/L_c	7.45	1.224	1.075	1.021	1.0046	between 1 and $1 + 0.5\,(a/R)^2$.

The expression L_c is more accurate than one might expect. It gives a reasonable approximation already for $R = 3a$, and the error roughly decays like $O(a^2/R^2)$.

Rectangle

This case is more complicated because in principle also the shape of the corners comes into play (curvature radius?). But the simplest thing to do is to evaluate the curve integral (4) for a rectangle with edges of length c and d. Orthogonal edges decouple because of the scalar product $\mathbf{j} \cdot \mathbf{j}'$, and the first contribution are the terms (7) for the four edges by themselves. The second contribution are the parts of the curve integral (4) with $\mathbf{x}$ on one edge and $\mathbf{x}'$ on the opposite one. The condition $|s - s'| > b$ is irrelevant for these cross terms, and one easily obtains for parallel edges of length c and distance d

$$L_c(c, d) = \frac{\mu_0}{4\pi} \left(4\sqrt{c^2 + d^2} - 4d - 4c \operatorname{asinh}(c/d) \right),$$

and the sum together with the Y-term of equation (4) is

$$\begin{aligned} L_c \;&=\; \frac{\mu_0}{\pi} \{\, c \ln \frac{2c}{a} + d \ln \frac{2d}{a} - (c + d)(2 - Y) \\ &\quad + 2\sqrt{c^2 + d^2} - c \operatorname{asinh}(c/d) - d \operatorname{asinh}(d/c) + a \,\}. \end{aligned} \tag{8}$$

This expression also may be found in the literature, with sometimes a factor 2 at the a-term.[3] The table displays the ratio of the numerically evaluated self-inductance L of a square with border length c and corners with curvature radius a and the curve integral L_c for different border length c,

c/a	5	10	20	40	80	160
L/L_c	1.1028	1.0284	1.0096	1.00363	1.00145	1.00061
$(L-L_c)/\mu_0$	0.306	0.285	0.2756	0.2706	0.2673	0.2673
L/L_c^{exact}	1.0055	1.0059	1.0017	1.00121	1.00052	1.00022

The curvature radius a is minimal in that the centre of curvature lies on the inner border of the wire. It is remarkable that the absolute error nearly remains constant. The last row of the table displays the ratio of the exact self inductance and the exact curve integral (4) (with round corners), also evaluated numerically. This expression is a better approximation for small c/a, where the square with round corners degenerates to a ring.

Equilateral triangle

The curve integral (6) for an equilateral triangle with edge length c consists of three times the expression (4) for the edges by themselves and three times the interaction energy $L_c(c, c, 120)$ of adjacent edges (with s on one edge and s' on the other, see appendix D. There is no such interaction for rectangles because of the scalar product),

$$L_c = \frac{\mu_0}{2\pi} 3c \left\{ \ln\left(\frac{c}{a}\right) - 1 - \ln\frac{3}{2} \right\}.$$

The table displays the ratio of the exact self-inductance L of an equilateral triangle with border length c and corners with curvature radius a and the curve integral L_c for different border length c,

c/a	5	10	20	40	80	160
L/L_c	2.77	1.20	1.057	1.020	1.0076	1.0031
$(L-L_c)/\mu_0$	0.8605	0.863	0.8646	0.8675	0.8692	0.884

The absolute error $(L-L_c)/\mu_0$ is nearly constant also here.

Parallel wires

For a loop consisting of arbitrary long parallel wires the condition $a \ll l$ is perfectly fulfilled and the curve integral gives the exact self inductance even for minimal distance $d = 2a$,

$$L_c = \frac{\mu_0 l}{\pi} \left(\ln\frac{d}{a} + Y \right).$$

This expression is the limiting case of the expression (8) for a long rectangle. The point is, that the contribution of the corners becomes negligible for a long rectangle, and that the replacement of the (circular symmetric) current by a line current doesn't change the magnetic field according to Ampere's law. Of course, the assumption of a circular symmetric current distribution gets wrong in the skin effect case if the wires are close to each other because of additional screening currents.

To summarize, formula (6) is rather accurate even for circuits with a linear extension as small as 20 times the wire radius, even if the circuit contains sharp corners.

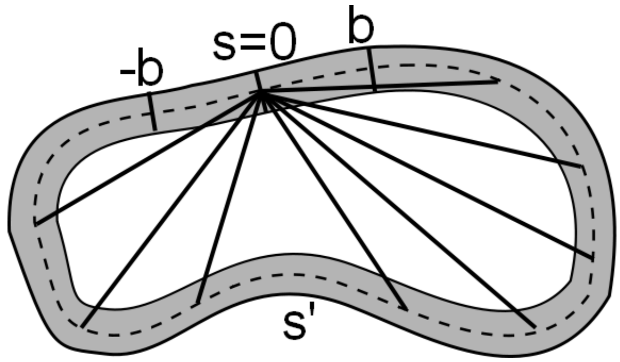

Figure 2: A wire loop and distances relevant for calculating the energy of a disc at $s=0$ relative to other points in the wire.

4 Error estimation

According to equation (5) the self inductance may be written as $L=(\mu_0/4\pi)\left(\overline{L}+\widehat{L}\right)$, where $\widehat{L}$ contains the short segments and $\overline{L}$ the complement. The self inductance L of course doesn't depend on the arbitrary segment length b.

Let us now introduce some notation. We use a coordinate system $\{s,r,\phi\}$ in the wire where the length s along the wire axis is cyclic with period l, and the coordinates $\{r,\phi\}$ describe planes perpendicular to the wire axis. The intersections of the planes and the wire are assumed to be circular, $0\leqq r\leqq a$ and $0\leqq\phi\leqq 2\pi$. The volume element reads $dV=(1+r\cos\phi/R(s))\,rdrd\phi ds$, where $R(s)$ denotes the curvature radius of the wire, and $\phi=0$ at the outer border of the wire (this is possible at least locally). The volume element also may be written as $dV=dsd\widetilde{A}$ with $d\widetilde{A}=(1+r\cos\phi/R(s))\,dA$ and area element $dA=rdrd\phi$. The coordinates become cylindrical coordinates for straight wire segments, i.e. for $R=\infty$.

The current density $\mathbf{j}$ is normalized, that is $\int dA\,|\mathbf{j}|=\int d\widetilde{A}\,|\mathbf{j}|=1$. We will also need the radial moment

$$a_2=\left\langle r^2\right\rangle=\int d\widetilde{A}r^2\,|\mathbf{j}|$$

of the current distribution. In the skin effect case of course $a_2=a^2$.

One quantity of interest then is

$$\overline{L}(0)=\int ds'\theta(|s'|-b)\int d\widetilde{A}d\widetilde{A}'\frac{\mathbf{j}(0,r,\phi)\cdot\mathbf{j}(\mathbf{x}')}{|\mathbf{x}(0,r,\phi)-\mathbf{x}(s',r',\phi')|},\tag{9}$$

the energy of the current in the plane at $s=0$ with respect to the current at $|s'|>b$, see figure (2). The symbol θ denotes the step function. To obtain $\overline{L}$ from $\overline{L}(0)$ requires to move the "reference point" $s=0$ around the wire, that is to integrate over s. The curve integral

$$\overline{L}_\gamma(0)=\oint ds'\theta(|s'|-b)\frac{\cos(\mathbf{j}(0),\mathbf{j}(s'))}{|\mathbf{x}(0,0,0)-\mathbf{x}(s',0,0)|},\tag{10}$$

is an approximation for $\overline{L}(0)$.

Similarly we write the short segment around the plane at $s=0$ as

$$\widehat{L}(0)=\int ds'\theta(b-|s'|)\int d\widetilde{A}d\widetilde{A}'\frac{\mathbf{j}(0,r,\phi)\cdot\mathbf{j}(0,r',\phi')}{|\mathbf{x}(0,r,\phi)-\mathbf{x}(s',r',\phi')|},\tag{11}$$

These definitions allow to write

$$\frac{4\pi}{\mu_0}L(0) = \overline{L}(0) + \widehat{L}(0) = \tag{12}$$
$$= \left\{\left(\overline{L}_\gamma + \widehat{L}_\gamma\right) + \left(\overline{L} - \overline{L}_\gamma\right) + \left(\widehat{L}_0 - \widehat{L}_\gamma\right) + \left(\widehat{L} - \widehat{L}_0\right)\right\}_{s=0},$$

where we have added and subtracted the curve integral $\overline{L}_\gamma(0)$, the segment integral $\widehat{L}_0(0)$ for a straight segment from equation (A.4) and the approximation $\widehat{L}_\gamma(0) = 2(\ln(2b/a) + Y)$ for $\widehat{L}_0(0)$.

The first bracket in equation (12) now is formula (6), the second bracket contains the difference of the volume and the curve integral (for $|s'| > b$), the third bracket is a simple power series in a/b, and the fourth bracket is the difference of the actual segment integral (11) and the segment integral for a straight segment. It is evident that the error becomes small in suitable limits, and we now want to determine the order of magnitude of the error.

Current loops consisting of straight segments

We first consider a current loop consisting of straight segments, each with a length comparable with the total length l, and a finite number of corners, each with a length comparable with the wire radius a. We show that the error is of order a. The procedure is simple in principle but circumstantial.

The first error term of equation (12), $\overline{L}(0) - \overline{L}_\gamma(0)$, contains a volume current in one direction and an opposite current of same strength on the wire axis. The reference point $s = 0$ mostly will lie on a straight segment, a distance $c > b$ away from the end of the segment. In the notation of appendix A the contribution of the segment becomes

$$\Delta\overline{L}(0) = \int_b^c ds'dAdA'\left(\frac{1}{\sqrt{N^2+s'^2}} - \frac{1}{s'}\right)\mathbf{j}(r)\,\mathbf{j}(r') = \int dAdA'\; A_1\left(\frac{N}{s'}\right)\Bigg|_b^c \mathbf{j}(r)\,\mathbf{j}(r')$$
$$= \left(\frac{1}{c^2} - \frac{1}{b^2}\right)\frac{\langle N^2\rangle}{4} - \left(\frac{1}{c^4} - \frac{1}{b^4}\right)\frac{3\langle N^4\rangle}{32} + \ldots$$

The functions A_1 and N also are defined in appendix A, here we only need $\langle N^k\rangle \sim a^k$. An inspection of equation (A.4) shows that the second error term $\widehat{L}_0 - \widehat{L}_\gamma$ of equation (12) exactly cancels the b terms in $\Delta\overline{L}(0)$ above (the additional factor 2 in $\widehat{L}_0 - \widehat{L}_\gamma$ is needed for the same purpose on the other side of the reference point). This cancellation isn't by accident, it comes about because $\overline{L}$ and $\widehat{L}_0$ from first and second error term are adjacent integrals of the same type.

To get the actual error requires to integrate over the reference point,

$$\Delta\overline{L} = \int_b^l \left(\frac{\langle N^2\rangle}{4c^2} - \frac{3\langle N^4\rangle}{32c^4} + \ldots\right)dc = \frac{\langle N^2\rangle}{4b} - \frac{\langle N^4\rangle}{32b^3} + \ldots + O\left(a^2/l\right) = O(a).$$

Here we now have selected a length b of order $O(a)$ and neglected a^2/l against a.

The other straight segments of the current loop also contribute to the first error term $\overline{L}(0) - \overline{L}_\gamma(0)$, but are away at least a distance $c > b$ from the reference point. It is shown in appendix B that straight segments with opposite currents in the volume and in the axis of the wire are equivalent to multipoles at both ends with a potential of order

$a^2/|\mathbf{x}|^2$. Integrating over the reference point thus completely analogously leads to an error $\Delta\overline{L} = O(a)$. It is even better, the corners (also with opposite currents) also are equivalent to multipoles of the same type and order of magnitude (this can be verified for instance from the dipole term in the next section).

One thus may combine the ends of the straight segments and the corners to one multipole and claim that the error $\left(\overline{L} - \overline{L}_\gamma\right) = O(a)$ in equation (12) is caused by the corners. It is essential for this argument, that the integrals over the reference point converge, and the corners thus add independent error contributions.

What if $s = 0$ comes to lie on a corner? This only occurs for a lenght of order a in the s-integral and only contributes another error of the same order of magnitude.

It remains to examine the last error term, the fourth bracket of equation (12), involving segments of length $2b = O(a)$. This term differs from 0 only if s is within a corner. The s integral thus yields something like $O(a)\,\widehat{L}_0 = O(a)$.

To summarize, the leading error of formula (6) in the straight segment case is of order $\mu_0 a$, much smaller than the self inductance of order $\mu_0 l \ln(l/a)$ even for moderately large l/a. The error grows with the number of corners and depends on the shape of the corners, but doesn't depend on the size of the loop. This perfectly agrees with the numerical results for the square and the triangle above. The leading error presumably simply is the sum of contributions from the corners, each a functional of order $O(\mu_0 a)$ of its shape. In practice it normally suffices to know the order of magnitude of the error.

Smooth current loops

Another special case are smooth current loops with a minimal curvature radius R_0 of the same order of magnitude as the loop length l. The simplest example is the circular loop. There are no corners in this case, which were identified in the previous section as error source for loops consisting of straight segments and corners. So one may expect a smaller error here. To identify the order of magnitude of the error for smooth loops requires more effort.

The coordinates $\mathbf{x}$ and $\mathbf{x}'$ in the integral (9) above may be expanded like $\mathbf{x} = \mathbf{x}(s,0,0) + \mathbf{x}_1(s,r,\phi)$, and thus

$$\begin{aligned} \mathbf{x} - \mathbf{x}' &= \mathbf{x}_{s,s'} + \mathbf{x}_1 - \mathbf{x}_1', \\ (\mathbf{x} - \mathbf{x}')^2 &= x_{s,s'}^2 + 2\mu x_{s,s'} + \nu^2 \end{aligned}$$

with $\mathbf{x}_{s,s'} = \mathbf{x}(s,0,0) - \mathbf{x}(s',0,0)$ the distance of the projections onto the axis and $|\mathbf{x}_1|$ and $|\mathbf{x}_1'|$ of order $O(a)$. The abbreviations are

$$\begin{aligned} \mu &= \widehat{\mathbf{x}}_{ss'} \cdot \mathbf{x}_1 + \widehat{\mathbf{x}}_{s's} \cdot \mathbf{x}_1', \\ \nu^2 &= (\mathbf{x}_1 - \mathbf{x}_1')^2 . \end{aligned}$$

The procedure now is to use the multipole expansion

$$\begin{aligned} \frac{1}{|\mathbf{x} - \mathbf{x}'|} &= \frac{1}{x_{s,s'}} - \frac{\mu}{x_{s,s'}^2} + \frac{1}{2x_{s,s'}^3}\left(3\mu^2 - \nu^2\right) - \frac{1}{2x_{s,s'}^4}\left(5\mu^3 - 3\mu\nu^2\right) \\ &\quad + \frac{1}{8x_{s,s'}^5}\left(35\mu^4 - 30\mu^2\nu^2 + 3\nu^4\right) + O\left(a^5 x_{s,s'}^{-6}\right). \end{aligned} \tag{13}$$

This expansion converges for $x_{s,s'} \geqq |\mathbf{x}'_1 - \mathbf{x}_1|$, the coefficients are the coefficients of the Legendre polynomials. Inserting the leading *monopole* term $1/x_{s,s'}$ of equation (13) into equation (9) reproduces the curve integral $\overline{L}_\gamma(0)$. The higher multipole terms describe the difference between the volume integral $\overline{L}(0)$ and the curve integral. The s'-integral in the multipole terms converges and the difference thus again is mainly a local quantity. The nominal order of magnitude of the multipole terms $\overline{L}_n$ is a^n/b^n, $n \geqq 1$. With a length $b = \sqrt{aR}$ the order of magnitude becomes $(a/R)^{n/2}$, and the expansion up to the hexadecupole ($n = 4$) is needed to get the $(a/R)^2$ approximation.

It is expedient to use a coordinate-free notation for the factor $(1 + r\cos\phi/R(s))$ in the volume element,

$$(1 + r\cos\phi/R(s))\, r dr d\phi ds = \left(1 + \mathbf{x}_1 \cdot \mathbf{R}(s)/R^2(s)\right) r dr d\phi ds,$$

where $\mathbf{R}(s)$ is the local curvature radius vector. Integrals over ϕ and ϕ' then may be evaluated with the help of

$$\begin{aligned} \langle \mathbf{x}_1 \rangle &= 0, \\ \langle (\mathbf{x}_1)_m (\mathbf{x}_1)_n \rangle &= r^2 P_{m,n}/2, \end{aligned}$$

where $P_{m,n}$ is the projection operator projecting onto the plane perpendicular to the wire axis. This implies $\mathbf{P} \cdot \mathbf{R} = \mathbf{R}$.

Inserting now the multipole expansion (13) into the volume integral (9) generates an expansion of the difference of the volume and the curve integral in the region $|s'| > b$.

Dipole A contribution to $\overline{L}(0)$ only comes from the combination of the $\mathbf{x}_1$ from μ with the $\mathbf{x}_1$ from the volume element, and vice versa for $\mathbf{x}'_1$,

$$\overline{L}_1(0) = -\frac{a_2}{2} \oint ds' \theta(|s'| - b) \left(\frac{\widehat{\mathbf{x}}_{0,s'} \cdot \mathbf{R}}{R^2} + \frac{\widehat{\mathbf{x}}_{s',0} \cdot \mathbf{R}'}{R'^2} \right) \frac{\cos\alpha}{x_{0,s'}^2}.$$

The angle α denotes the angle between the current at s and the current at s'. The integral over s' converges for large s' because of $x_{0,s'} \sim s'$ and instead of the nominal a/b there at most remains $a^2/(bR)$. An interesting special case is a segment with $R = \infty$ most of the time and $R = O(a)$ along a length s' also of order $O(a)$. The s' integration then reproduces the contribution $\overline{L}_1(0) = O\left(a^2/|\mathbf{x}|^2\right)$ from a multipole at the position of a corner used above.

For a smooth curve near $s' = 0$ approximately $\widehat{\mathbf{x}}_{0,s'} \cdot \mathbf{R} = \widehat{\mathbf{x}}_{s',0} \cdot \mathbf{R}' = s'/2$. The integral therefore is logarithmic and not local any more. This means that the error depends on the global shape of the curve. But the logarithmic factor changes only slowly and the error gets the order of magnitude $(a^2/R^2)\ln(R/b) \sim (a^2/R^2)\ln(R/a)$.

Quadrupole After integration over the angles ϕ and the radial coordinates r there remains

$$\overline{L}_2(0) = a_2 \oint ds' \theta(|s'| - b) \left(\frac{3}{4}\left(2 - (\widehat{\mathbf{n}} \cdot \widehat{\mathbf{x}}_{0,s'})^2 - (\widehat{\mathbf{n}}' \cdot \widehat{\mathbf{x}}_{0,s'})^2\right) - 1 \right) \frac{\cos\alpha}{x_{s,s'}^3},$$

where $\widehat{\mathbf{n}}$ denotes a unit vector in the direction of the wire axis and the expression $P = 1 - \widehat{\mathbf{n}}\widehat{\mathbf{n}}$ for the projection operator was used. In $\overline{L}_2(0)$ one may recognize $1 - (\widehat{\mathbf{n}} \cdot \widehat{\mathbf{x}}_{0,s'})^2 = \sin^2\psi$

and $1-\left(\widehat{\mathbf{n}}'\cdot\widehat{\mathbf{x}}_{0,s'}\right)^2=\sin^2\psi'$, where ψ denotes the angle between the distance vector $\mathbf{x}_{0,s'}$ and the wire axis. With $\psi'=\psi=2\alpha=s'/2R$ for a smooth current loop there remains an error $(a^2/R^2)\ln(R/b)$. The $-a_2/b^2$ term (from the -1) is peculiar. With the choice $b=\sqrt{aR}$ it would be of order a/R, but it gets cancelled by the third error bracket $\widehat{L}_0-\widehat{L}_\gamma$ of equation (12), see also equation (A.2) and (A.4).

Oktupole The oktupole contributes at most a term of order a^3/b^3. But the $\cos\phi$ and $\sin\psi$ from the odd power of μ make the actual contribution smaller than the expected $\sim a^2/R^2$.

Hexadecupole The hexadecupole term is of order $a^4/b^4\sim a^2/R^2$ and its leading part must be kept. The μ factors contain a factor $\sin\psi\sim b/R$ and may be dropped. The ν^4 leads to

$$\overline{L}_4(0)=\frac{3}{8}\oint ds'\theta\left(|s'|-b\right)\left[2a_4+a_2^2\left(3+\cos^2\alpha\right)\right]\frac{\cos\alpha}{x_{0,s'}^5}.$$

The factors $\cos\alpha$ may be replaced with 1 because of $\alpha=s'/R$. The integral converges and contributes an error like a^2/R^2. But this term also gets cancelled by the third error bracket $\widehat{L}_0-\widehat{L}_\gamma$ of equation (12), see also equation (A.4). To be more explicit, the $A_1(N/b)$ gives an N^4 which gives $2a_4+4a_2^2$ after averaging over the angle ϕ.

For smooth current loops it finally follows

$$\frac{4\pi}{\mu_0}L=\left(\overline{L}_\gamma+\widehat{L}_\gamma\right)+O\left(\left(a^2/R\right)\ln(R/b)\right)+\left(\widehat{L}-\widehat{L}_0\right), \tag{14}$$

where the first bracket on the r.h.s is formula (6), evaluated with $b=\sqrt{aR}$. The remaining segment integral $\widehat{L}$ for a segment with curvature radius R is evaluated in the appendix, the result (C.1) contributes another logarithmic error $(a^2/R^2)\ln(R/b)$, and an error b^2/R^2. But the latter term only cancels the b-dependence of the curve integral $\overline{L}_\gamma$, evaluated as (C.2) also in appendix C. As was argued in the simple derivation of the formula above the cancellation comes about, because b is the lower limit in the curve integral and the upper limit in the volume integral.

To recapitulate, for segment length $b=\sqrt{aR}$ there remain formula (6) and small terms of order $O\left((a^2/R)\ln(R/a)\right)$.[1] We now drop these small terms, since they only contribute to the expected error. There is no small lenght scale in the formula $L_\gamma+\widehat{L}_\gamma$ and its b-dependence is under control for all $b<\sqrt{aR}$ and given by (C.2). It contains a "large" $b^2/R^2\sim a/R$ term, which gets cancelled by (C.1). In total there only remains $\overline{L}_\gamma+\widehat{L}_\gamma$ *without* its "large" a/R-term. But this is *identical* with $\overline{L}_\gamma+\widehat{L}_\gamma$ evaluated for $b=a/2$, apart from another contribution to the a^2/R^2 error, see (C.2). In other words, the simple formula (4) approximates self inductance with an error of order $(a^2/R)\ln(R/b)$ or a^2/R.

5 Conclusions

The curve integral (4) for the self inductance of a wire loop is only a little bit more complicated than the Neumann formula for the mutual inductance of two wire loops.

[1] The error terms may be evaluated analytically to order a^2/R^2 for a cicular loop. The result agrees with the result of Rayleigh and Niven in the constant current case (see ref. [4]) and perfectly agrees with the numeric results above in the skin effect case.

The exact expression for self inductance is a 6-dimensional integral with a logarithmic divergence and several length scales. Nevertheless clear statements follow for the accuracy of formula (4), for loops consisting of straight segments as well as for smooth loops. The error originates from the curved parts of the loop, and is of order $\mu_0 a$ or $\mu_0 a^2/l$, negligible for most practical purposes. The leading error presumably even might be given as a sum over the corners for loops consisting of straight segments or as additional curve integrals for smooth current loops (see the dipole and quadrupole contributions above). The techniques used for error estimation also might be used for the Neumann formula.

The self inductance curve integral can be evaluated analytically in many cases, for instance for current loops consisting of coplanar straight segments. But also the numerical evaluation of a two-dimensional integral with a computer program is a breeze with appropriate numerical libraries.

Current distributions which are not circular symmetric also lead to formula (4), with a cutoff and a constant Y depending on the current distribution. An example are circuits consisting of coplaner flat strips of width w. The self inductance of such circuits is given in the accuracy described above by formula (4) with $a = w$ and $Y = 3/2$.

A Contribution from straight segments of length $2b$

The contribution $\widehat{L}$ to the self inductance in equation (5) is due to the interaction of the current in the plane s with the current in all planes s' with $|s' - s| < b$. This value depends on the current distribution in the wire and on the wire geometry within the segment $[s - b,\, s + b]$, but may be evaluated if the segment is straight or slightly curved. This s-dependent value of course still is to be integrated over all s.

To get an approximation for $\widehat{L}$ in the straight wire case use cylinder coordinates with a length s along the axis and area element $dA = r dr d\phi$ (see figure (1)). This leads to

$$\begin{aligned} \widehat{L}_0 &= \oint ds \widehat{L}(s), \\ \widehat{L}_0(s) &= \left(\int \frac{\mathbf{j}(r)\,\mathbf{j}(r')}{|\mathbf{x}(s,r,\phi) - \mathbf{x}'|} ds' dA' dA \right)_{|s(\mathbf{x}') - s| < b}. \end{aligned} \tag{A.1}$$

In the latter integral $\mathbf{x}$ extends over the plane through the centre of a cylindrical segment, $\mathbf{x}'$ extends over the complete segment. The integral $\widehat{L}_0(s)$ of course is independent of s. The integral over s' (from $-b$ to b) may be performed using $|\mathbf{x} - \mathbf{x}'|^2 = N^2 + s'^2$, $N^2 = r^2 + r'^2 - 2rr'\cos(\phi - \phi')$,

$$\begin{aligned} \widehat{L}_0(0) &\cong 2 \int dA dA' \mathbf{j}(r)\,\mathbf{j}(r') \operatorname{asinh}(b/N) \\ &= 2 \int dA dA' \mathbf{j}(r)\,\mathbf{j}(r') \left\{ \ln(2b/a) - \ln(N/a) + A_1(N/b) + ... \right\} \end{aligned}$$

In the second line the expansion

$$\begin{aligned} \operatorname{asinh}(x) &= \ln(2x) + A_1(1/x) \\ A_1(x) &= \sum\nolimits_{n=1}^{\infty} \frac{1 \cdot 3 ... (2n-1)}{2 \cdot 4 ... 2n} \frac{(-1)^{n+1}}{2n} x^{2n} = x^2/4 - 3x^4/32 + ... \end{aligned} \tag{A.2}$$

was used. The expansion converges because of $N = O(a) \ll b$. It doesn't matter which ϕ' occurs in the ϕ-integral and thus we now set $\phi' = 0$. Because of $\int dA\,|\mathbf{j}| = 1$ the leading term simply becomes $2\ln(2b/a)$. The second term follows from $\ln(N/a) = \frac{1}{2}\ln(\rho^2 + \rho'^2 - 2\rho\rho'\cos\phi)$ with $\rho = r/a$ and $\rho' = r'/a$ and

$$\frac{1}{2\pi}\int_0^{2\pi} \ln\left(\rho^2 + \rho'^2 - 2\rho\rho'\cos\phi\right) d\phi = 2\ln(\rho_>)\,, \tag{A.3}$$

where $\ln(\rho_>) = \theta(\rho - \rho')\ln\rho + \theta(\rho' - \rho)\ln\rho'$. This term thus vanishes in the skin effect case where the current differs from 0 only for $\rho = \rho = 1$. The current density in the constant current case is $j(r) = 1/(\pi a^2)$ and the second term becomes

$$-\frac{(2\pi)^2}{\pi^2}2\int_0^1 d\rho\rho\int_0^1 d\rho'\rho'\ln(\rho_>) = 1/2.$$

A good approximation for $b \gg a$ thus is

$$\begin{aligned}\widehat{L}_0(0) &= \widehat{L}_\gamma(0) + O\left(a^2/b^2\right), \\ \widehat{L}_\gamma(0) &= 2\ln(2b/a) + 2Y\end{aligned} \tag{A.4}$$

with $Y = 1/4$ for a constant current distribution and $Y = 0$ in the skin effect case. The trivial integration over s leads to $\widehat{L}_0 \cong l\widehat{L}_\gamma(0)$.

B Semi-infinite straight segments

One error term in equation (12) is the difference of the volume integral $\overline{L}(0)$ and the curve integral $\overline{L}_\gamma(0)$. For current loops consisting of straight segments and sharp corners this difference contains the potential of straight segments of length c with a current in the bulk of the wire and an opposite current on the axis. These currents are collinear and scalar potential theory applies. This allows to generate the potential of the segment by subtracting the potential of a semi-infinite current distribution of this type from the potential of an identical semi-infinite current distribution shifted by length c along the axis. The essential result is that the potential of a straight segment of this type is generated by multipoles at the ends of the segment.

The potential of a semi-infinite segment in cylinder coordinates is

$$\begin{aligned}\phi(r,z) &= \int_{-\infty}^0 \left\langle \left(N^2 + (z'-z)^2\right)^{-1/2} - \left(r^2 + (z'-z)^2\right)^{-1/2} \right\rangle dz' \\ &= \left\langle \ln\frac{r}{N} + \operatorname{asinh}\frac{z}{r} - \operatorname{asinh}\frac{z}{N} \right\rangle.\end{aligned}$$

The segment extends along $z < 0$, $N^2 = r^2 + r'^2 - 2rr'\cos\phi'$, and the angular bracket $\langle ... \rangle$ is an abbreviation for $\int d\phi' r' dr' j(r') ...$ with $r' < a$. The current $j(r')$ is a unit current $\langle 1 \rangle = 1$. To verify that the potential concentrates around the end at $z = 0$ first consider the case $r = 0$ and $z > a > r'$. For $r \to 0$ one gets $N = r'$ and the above equation leads to (see equation (A.2))

$$\phi(r=0,z) = \left\langle \ln\frac{2z}{r'} - \operatorname{asinh}\frac{z}{r'} \right\rangle = \left\langle \frac{-r'^2}{4z^2} + ... \right\rangle = \frac{-\langle r'^2 \rangle}{4z^2} + O\left(a^4/z^4\right).$$

In the skin effect case of course $\langle r'^2 \rangle = a^2$. The potential thus decays like $1/z^2$ in the direction of the axis.

Next consider the cones $z = qr$, with $q \neq 0$ some constant and $r > a$,

$$\phi(r, qr) = \left\langle \frac{-1}{2} \ln\left(1 + \frac{r'^2}{r^2} - 2\frac{r'}{r} - 2\cos\phi'\right) + \operatorname{asinh} q - \operatorname{asinh} \frac{q}{\sqrt{1 + r'^2/r^2 - 2\cos(\phi')\, r'/r}} \right\rangle .$$

The integral $\int d\phi' \ln(...)$ vanishes (this corresponds to Gauss's law), and expanding the asinh in $1/r$ leads to

$$\phi(r, qr) = \frac{-q}{\sqrt{1+q^2}} \frac{a_2}{4r^2} + O\left(r^{-4}\right).$$

The potential thus decays in all directions like $a_2 |\mathbf{x}|^{-2}$, and a straight segment of finite length corresponds to two such multipoles, one at each end.

C Contribution from a curved segment

The goal is to evaluate the integral $\widehat{L}(0)$ from equation (11) for a segment of length $2b$ and constant curvature radius R,

$$\widehat{L}_R(0) = \int \frac{ds' d\widetilde{A} d\widetilde{A}'}{|\mathbf{x}(0, r, \phi) - \mathbf{x}'|} \theta(b - |s'|)\, j(r)\, j(r') \cos\left(\frac{s'}{R}\right).$$

The distance up to order $O(R^{-2})$ follows from

$$\begin{aligned} (\mathbf{x} - \mathbf{x}')^2 &\cong s'^2 + N^2 + q, \\ N^2 &= r^2 + r'^2 - 2rr' \cos(\phi - \phi'), \\ q &= s'^2 \left[(r'\cos\phi' + r\cos\phi) R^{-1} + rr' R^{-2} \cos\phi \cos\phi'\right] - s'^4/\left(12R^2\right) + s'^6/\left(360R^4\right). \end{aligned}$$

Expanding in q gives

$$\frac{1}{|\mathbf{x} - \mathbf{x}'|} = \frac{1}{(s'^2 + N^2)^{1/2}} - \frac{q}{2(s'^2 + N^2)^{3/2}} + \frac{3q^2}{8(s'^2 + N^2)^{5/2}} + ...$$

The N in the denominators is negligible for $b \gg a$ and the elementary integrals lead to

$$\widehat{L}_R(0) = \widehat{L}_0(0) - \frac{11}{24}\frac{b^2}{R^2} + O\left(\frac{a_2}{R^2} \ln \frac{b}{a}\right). \tag{C.1}$$

The term of order $O(a_2/R^2)$ contributes to the error of the self inductance formula with the expected order of magnitude. It is essential however, that the $O(b^2/R^2)$ term, which is of order a/R because of $b = \sqrt{aR}$, cancels against a contribution from the curve integral $\overline{L}_\gamma(0)$ from equation (10).

To verify this cancellation start with

$$\frac{d}{db}\overline{L}_\gamma(0) = \frac{-\cos(\mathbf{j}(0), \mathbf{j}(b))}{|\mathbf{x}(0) - \mathbf{x}(b)|} - \frac{-\cos(\mathbf{j}(0), \mathbf{j}(-b))}{|\mathbf{x}(0) - \mathbf{x}(-b)|}.$$

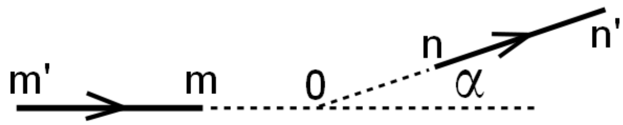

Figure 3: Two non-adjacent coplanar straight segments. The coordinates m, m' and n, n'measure the distance of the end points from the intersection of the segment extensions.

Inserting $|x-x'| = 2R\sin b/2R$ and $\cos(\mathbf{j}(0), \mathbf{j}(b)) = \cos b/r$ for a section with curvature radius R gives

$$\frac{d}{db}\overline{L}_\gamma(0) = \frac{-2}{b}\left(1 - \frac{11}{24}\left(\frac{b}{R}\right)^2 + ...\right),$$

with integral

$$\overline{L}_\gamma(0) = const - 2\ln\frac{2b}{a} + \frac{11}{24}\frac{b^2}{R^2} + ... \tag{C.2}$$

D Curve integral for adjacent straight segments

For completeness we display here the curve integral contribution to the self inductance from adjacent straight segments of length c and d with an angle α between the currents,

$$\begin{aligned} L_\gamma(c,d,\alpha) &= \frac{\mu_0}{2\pi}\cos(\alpha)\{c\,\mathrm{asinh}\,\frac{d + c\cos\alpha}{c\sin\alpha} + d\,\mathrm{asinh}\,\frac{c + d\cos\alpha}{d\sin\alpha} \\ &\quad - (c+d)\,\mathrm{asinh}\,\frac{\cos\alpha}{\sin\alpha} - \frac{2b}{\sqrt{1(1-\cos\alpha)}}\,\mathrm{asinh}\,\frac{1-\cos\alpha}{\sin\alpha}\}. \end{aligned}$$

For each corner such a term is to be added to the contribution (7) of the (straight) segments by themselves. The b-term is of order $O(a)$ for $b = a/2$ and normally may be neglected.

E Curve integral for non-adjacent coplanar straight segments

What then is missing for calculating the self inductance of a loop consisting of arbitrary coplanar straight segments is the mutual contribution from non-adjacent straight segments, see figure (3).

$$\begin{aligned} L(m,m',n,n',\alpha) &= \frac{\mu_0}{2\pi}\{A(m',n',n,\alpha) + A(n',m',m,\alpha) + A(m,n,n',\alpha) + A(n,m,m',\alpha)\}, \\ A(w,u,v,\alpha) &= w\,\mathrm{asinh}\left(\frac{u + w\cos\alpha}{w\sin\alpha}\right) - w\,\mathrm{asinh}\left(\frac{v + w\cos\alpha}{w\sin\alpha}\right) \end{aligned}$$

This leads to an unwieldy expression already for a hexagon, but the calculation of the self inductance of such loops is a matter of algebra and geometry.

References

[1] J.D. Jackson, "Classical Electrodynamics", p. 261. Wiley 1975.

[2] F. E. Neumann, "Allgemeine Gesetze der inducirten elektrischen Ströme". Abhandlungen der Königlichen Akademie der Wissenschaften zu Berlin, aus dem Jahre 1845: 1-87, 1847.

[3] E.B. Rosa, The self and mutual inductances of linear conductors. Bulletin of the Bureau of Standards 4 (2) (1907) 301-344.

[4] E.B. Rosa and F.W. Grover, Formulas and tables for the calculation of mutual and self-inductance. Scientific papers of the bureau of standards (1916). Government printing office, Washington 1948.